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LETTER TO THE EDITOR

Interfaces in a random medium and replica symmetry breaking

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Abstract. We develop a variational approach for studying interfaces and other manifolds in a disordered quenched medium. The method may be applied to problems which range from directed polymers to the interface of an Ising model in a random magnetic field. We find that replica symmetry is spontaneously broken and the results of the Flory approximation are recovered in a simple way. Corrections to this approximation may be computed in a systematic way.

The behaviour of fluctuating manifolds in a random medium (quenched disorder) is very interesting and its understanding has many applications to different fields of physics (Nattermann and Rujan 1989, Halpin-Healey 1989); in spite of the serious efforts which have been devoted to the study of this problem the situation is still confused and conflicting results have been obtained (Efetov and Larkin 1977, Kogan and Wallace 1981, Fisher 1986, Brezin and Orland 1986). The aim of this letter is to show that a crucial ingredient, replica symmetry breaking (Mézard *et al* 1987), was lacking in previous analysis and that, after its introduction, we recover reasonable results in a simple way.

In a nutshell the problem may be formulated as follows. In the continuum limit the Hamiltonian is

$$H = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \sum_{\mu=1,d} [\partial_\mu \omega(x)]^2 + \eta(x, \omega(x)) \right\} \tag{1}$$

where $\omega(x)$ is a vector with N components. The vector-valued function $\omega(x)$ is defined on a d-dimensional space and represents the coordinates of a d-dimensional manifold in a $(N+d)$ dimensional space. The x's parametrize the manifold which has coordinates $(x, \omega(x))$.

The function $\eta(x, y)$ (y being an N-dimensional vector) represents the effect of the disorder and is a quenched variable, which is usually supposed to be Gaussian distributed (the effects of having a non-Gaussian distribution for η are discussed in Zhang 1990). Different models may be obtained by choosing different forms for the correlation of the noise. In this letter we consider only the case where

$$\overline{\eta(x_1, y_1)\eta(x_2, y_2)} = (g/\lambda) N^{1+(\lambda/2)} \delta(x_1 - x_2)|y_1 - y_2|^{-\lambda} \tag{2}$$

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g being the coupling constant; when $\lambda = N$, the RHS of (2) becomes proportional to $\delta(y_1 - y_2)$ and the noise is short range (Medina *et al* 1989, Halpin-Healey 1989). The models we consider are thus characterized by three parameters (d, N and λ); for some values of these parameters the model may be not defined in the continuum and a cut-off is present (we only consider the case where $\lambda \geqslant -2$).

In the low-temperature limit the problem reduces to finding the solution of the stochastic differential equation

$$\Delta\omega(x) = \partial\eta(x, y)/\partial y|_{y=\omega(x)}. \tag{3}$$

It is crucial that, if there is more than one solution to (3), we should choose the one with lowest energy.

The long-range behaviour of these models at low temperatures is characterized by a single exponent ζ, which describes the growth of the transverse fluctuations of the manifold as function of the distance, i.e.

$$\langle[\omega(x) - \omega(y)]^2\rangle \propto |x-y|^{2\zeta} \tag{4}$$

for large $|x-y|$. Using scaling invariance and Galilean invariance the values of the other exponents may be related to ζ (Kardar and Zhang 1987).

There is a general agreement on the fact that, if the dimension d of the manifold is greater than 4, the exponent ζ (at least for not too large coupling g) is zero and it becomes positive as soon as d is less than 4. There is no consensus on the values of ζ in this last region. Indeed, in pertubation theory one finds the baffling result (Efetov and Larkin 1977, Brezin and Orland 1986)

$$\zeta = (4-d)/2 \tag{5}$$

independent of N or λ.

Similar results can be obtained by using supersymmetric arguments (Parisi and Sourlas 1979, Kogan and Wallace 1981).

However both pertubation theory and supersymmetry (Parisi 1984 and 1987) implicitly assume the existence of only one solution to (3). The presence of many solutions to this equation has been recognized to be the crucial ingredient which leads to the failure of (5) (Villain and Semeria 1983, Engel 1985), as can be explicitly seen in a zero-dimensional model (Schulz *et al* 1988).

Simple arguments based on naive dimensional counting lead to the Flory-type result (Villain 1982, Grinstein and Ma 1983):

$$\zeta = (4-d)/(4+\lambda). \tag{6}$$

Although equation (6) is not always the exact result, it certainly makes more sense than equation (5) (at least ζ depends on the form of the noise). Unfortunately we do not at present, have a satisfactory derivation of (6) for any value of the parameters; indeed the proof based on the so-called functional renormalization group (Fisher 1986) is technically incorrect (Brezin and Orland 1986) and it does not take into account the crucial physical point, i.e. the existence of many solutions of (3).

It seems natural to suppose that the interface has many equilibrium points (Villain and Semeria 1983); if this happens, standard perturbation theory is inadequate to describe fluctuations between free energy minima which are far from each other in phase space. The correct formalism to describe the system consists of using the replica approach with broken replica symmetry. This proposal seems to be quite natural as far as explicit computations for $d=1$ (directed polymers) strongly point towards the spontaneous breaking of the replica symmetry (Derrida and Spohn 1988, Parisi 1990b and Mézard 1990).

In this letter we will use a variational approach (in the spirit of Shakhnovich and Gutin 1989), which should be exact in the limit of large N. The solution with broken replica symmetry can be found explicitly and it leads to (6).

The first step consists in using the replica formalism and in integrating over the Gaussian noise η. We introduce n copies of the N-dimensional vector ω, which now carries two indices: ω_a^α, $\alpha=1, N$, $a=1, n$. The expectation value of the partition function to the nth power can be obtained by using the following Hamiltonian:

$$H_R=\int \mathrm{d}^d x\left\{\beta \sum_{a=1,n,\mu=1,d} \tfrac{1}{2}[\partial_\mu \omega_a(x)]^2\right.$$
$$\left.-(g/\lambda)\beta^2 N^{(1+\lambda/2)} \sum_{a=1,n;b=1,n}|\omega_a(x)-\omega_b(x)|^{-\lambda}\right\} \tag{7}$$
$$|\omega_a-\omega_b|^2 \equiv \sum_{\alpha=1,N}[\omega_a^\alpha-\omega_b^\alpha]^2.$$

Apart form the simple case where $\lambda=-2$ (Parisi 1990a), the model is not explicitly soluble and the interaction is not polynomial. It is interesting to note that the model is invariant under the group $\mathrm{O}(N)$ (i.e. rotations in physical space), under the S_n group (i.e. the permutation group of n replicas) and under the group of translations. The actual symmetry group is larger, but we do not need it. As usual, eventually we have to consider the limit n going to zero, which gives a distinctive flavour to the replica approach (Mézard *et al* 1987).

As usual a mean-field approach may be constructed by considering a class of Hamiltonians H_Q, whose partition function is computable as function of the parameters Q. The best Hamiltonian, which shall be the starting point in the mean-field approach, can be found as the solution of the variational problem for the free energy:

$$\partial F/\partial Q=0 \qquad F[Q]=\langle H_R\rangle_Q-S[H_Q] \tag{8}$$

where $\langle\ \rangle_Q$ denotes the expectation value with respect to the Hamiltonian H_Q and $S[H_Q]$ is the entropy of that Hamiltonian.

The simplest choice for H_Q is a quadratic functional of the field ω, which, without loss of generality, we can assume to be of the form:

$$H_Q=\int \mathrm{d}^d x \sum_{\alpha=1,N}\left\{\beta \sum_{a=1,n;\mu=1,d} \tfrac{1}{2}[\partial_\mu \omega_a^\alpha(x)]^2+\sum_{a=1,n;b=1,n} Q_{a,b}\omega_a^\alpha(x)\omega_b^\alpha(x)\right\}. \tag{9}$$

The matrix Q plays the role of a variational parameter; translational invariance implies that

$$\sum_a Q_{a,b}=0. \tag{10}$$

If more general quadratic forms of H_Q were considered, the variational principle always would produce a result of the form given in (9).

In momentum space the correlation function of two ω's is given by

$$G_{a,b}(k)=[1/(k^2+Q)]_{a,b}. \tag{11}$$

After a simple computation one finds that the free energy is proportional to

$$F \propto \text{constant} + \int \mathrm{d}^d k\{-k^2 \operatorname{Tr}[G(k)] + \operatorname{Tr}[\ln G(k)]$$

$$- gc/\lambda \sum_{a<b} \int \mathrm{d}^d k\{G_{a,a}(k) + G_{b,b}(k) - 2G_{a,b}(k)\}^{-\lambda/2} \tag{12}$$

where we have set $\beta = 1$ for simplicity and $c = N^{-\lambda/2}\Gamma(\frac{1}{2}N - \frac{1}{2}\lambda)/\Gamma(\frac{1}{2}N)$.

The mean-field equations (8) can be explicitly written as

$$Q_{a,b} = g\frac{c}{2}\left[\int \mathrm{d}^d k\{G_{a,a}(k) + G_{b,b}(k) - 2G_{a,b}(k)\}\right]^{-\gamma} \tag{13}$$

where $\lambda = 2\gamma - 2$.

The reader familiar with the diagrammatic approach will recognize in (13) the usual equation for the self-energy of a bosonic field, where tadpole graphs are considered in a self-consistent way. This equation is sometimes called the gap equation and the approach is denoted as the Hartree-Fock approximation. Usual diagrammatical (or functional) arguments tell us that, in the absence of loopholes due to infrared or ultraviolet divergences, (13) is exact when N goes to infinity and it can be used as the starting point of an expansion in powers of $1/N$.

The rest of this letter will be devoted to finding the solution of (13), assuming a hierarchical replica symmetry breaking scheme. We will assume that the matrix Q is given by the canonical form (Mézard *et al* 1987), as functional of the function $q(x)$, plus a term on the diagonal $\tilde{q}$, where $q(x)$ is defined in the same way as in spin glasses. The constraint of (10) translates into

$$\tilde{q} = \int \mathrm{d}x\, q(x). \tag{14}$$

If we try to solve (13) we face the problem of computing the propagator $G_{a,b}(k)$ in (11). This is again an ultrametric matrix which can be computed using techniques similar to Mézard and Parisi (1985). To the set $(\tilde{q}, q(x))$ we associate the function $\Omega(x)$, defined as

$$\Omega(x) = \tilde{q} - q(1) + \int_x^1 \mathrm{d}y\, y q'(y). \tag{15}$$

Then the function $g(x, k)$ and the diagonal term $\tilde{g}(k)$ corresponding to the matrix $G_{a,b}(k)$ are given by

$$\begin{aligned} \tilde{g}(k) - g(x, k) &= \frac{1}{x(k^2 + \Omega(x))} - \int_x^1 \frac{\mathrm{d}y}{y^2} \frac{1}{k^2 + \Omega(y)} \\ \tilde{g}(k) &= \frac{1}{k^2}\left(1 + \int_0^1 \frac{\mathrm{d}y}{y^2} \frac{\Omega y}{k^2 + \Omega y}\right) - q(0)/k^4. \end{aligned} \tag{16}$$

Equation (13) thus becomes, at small x,

$$q(x) = g\left(\int \mathrm{d}^d k[\tilde{g}(k) - g(x, k)]\right)^{-\gamma}$$

$$= g\left(\frac{c(d)}{x}\int_x^1 \Omega(x)^{1-\varepsilon} - c(d)\int_x^1 \mathrm{d}y/y^2\, \Omega(y)^{1-\varepsilon}\right)^{-\gamma} \tag{17}$$

where $d = 4 - 2\varepsilon$ and $c(d) = \pi^{d/2}\Gamma(1 - d/2)$.

Equation (17) is very similar to the one found by Shakhnovich and Gutin 1989 in a different context (self-interacting random heteropolymers).

If we differentiate twice with respect to x we find (neglecting multiplicative constants), the simple equation

$$(x-\Omega(x)^{\delta})q'(x)=0 \qquad \delta=1-\varepsilon\gamma/(1+\gamma). \tag{18}$$

Let us assume that for x near to zero $q'(x)$ is different from zero. We find that for small x, $q(x)$ behaves as $x^{(1/\delta)-1}$ and $q(0)=0$. The reasonable range for δ is the interval $0-1$. A value of δ greater that one should be interpreted as the absence of replica symmetry breaking, while a value of δ less then zero implies that the function $q(x)$ is identically zero for x less then a critical value, as can be shown by a detailed analysis. Replica symmetry is therefore broken in the region where the dimension is less than 4 and λ is greater than -2.

We can now compute the propagator $\tilde{g}(k)-g(x,k)$ where x is the distance between a and b in replica space; we find that the singular part for small x is given by

$$\int_x \mathrm{d}x' \frac{x'^{(1/\delta)-2}}{(k^2+x'^{(1/\delta)})^2} \tag{19}$$

which behaves as

$$\begin{aligned} &x^{(1/\delta)-1}/k^4 &&\text{for } k^2 \gg x^{1/\delta} \\ &x^{-(1/\delta)-1} &&\text{for } k^2 \ll x^{1/\delta}. \end{aligned} \tag{20}$$

The singular part of the diagonal part of the correlation can be obtained from (16).

We thus find that $\tilde{g}(k)$ behaves for small k as $k^{-2(1+\delta)}$. On the other hand the scaling law (3) implies that $\tilde{g}(k)$ behaves, for small k, as $k^{-d-\zeta}$; by comparing the two equations we find that

$$\begin{aligned} &\zeta=\varepsilon/(1+\gamma) &&\text{for } 1>\varepsilon\gamma/(1+\gamma) \\ &\zeta=\varepsilon-1 &&\text{for } 1<\varepsilon\gamma/(1+\gamma). \end{aligned} \tag{21}$$

Equation (21) is the Flory result (Halpin-Healey 1989), as can be easily checked.

The breaking of replica symmetry in a hierarchical fashion leads naturally to non-trivial critical exponents. The behaviour of the correlation function at large distance is related to the behaviour of the function $q(x)$ at small x. The physical implications of replica symmetry breaking will be discussed in more detail in Mézard and Parisi 1990; here we only note that the breaking of replica symmetry probably has deep consequences on the dynamics of the system and it is likely to be at the origin of the very slow approach to equilibrium in these systems.

The very nature of the variational approximation we have used strongly suggests that (21) cannot be exact for all values of N, while it is likely to be correct in the limit of infinite N, at least for γ in an appropriate range. The construction of an $1/N$ expansion seems to be feasible, although it may be quite involved. It is quite possible that the ε expansion is simpler. In any case the explicit computation of the first corrections to the mean-field approximation should be very instructive.

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